EXISTENCE AND UNIQUENESS OF SOLUTIONS TO STOCHASTIC HEAT EQUATIONS WITH MARKOVIAN SWITCHING AND FELLER PROPERTY

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ABSTRACT. In this paper, we focus on two-component Markov processes which consist of continuous dynamics and discrete events. Using the classical fixed point theorem for contractions to investigate the existence and uniqueness of solutions of stochastic heat equations with Markovian switching, then developing the corresponding the Feller property of the solution.

KEYWORDS. Markov chain, Markov process, mild solution, Feller property.

1. INTRODUCTION

Stochastic modelling plays a key role in sciences and industries. In particular, stochastic differential equations with Markovian switching (SDEs-MS in short) have a number of important applications, such as population dynamics, stochastic stabilization and financial modelling. For instance, Mao and Yuan [11] first systematically presented the theory of SDEs-MS and applications. Luo and Mao [9] showed that the positive solution of the stochastic population systems under regime switching did not explode in finite time with probability one. Moreover, Luo and Mao [10] demonstrated that the solution was stochastically ultimately bounded and average in time of the second moment of the solution was bounded. Settati and Lahrouz [13] investigated the existence of a unique ergodic stationary distribution of the stochastic n-dimensional population systems under regime switching. Zhang, Tong and Hu [17] applied SDEs-MS into stochastic interest rate model. They discussed the Feller property and unique stationary distribution for Cox-Ingersoll-Ross model with Markov switching. The limitation of previous studies only considered the stochastic systems in finite dimensions. However, Da Prato and Zabczyk [4, 5] introduced the stochastic equations in infinite dimensions, did not consider any Markov chain. They studied the existence and uniqueness of solutions of general stochastic evolution equations in Hilbert space, and investigated the qualitative properties of solutions. Recently, Anabtawi and Sathananthan [1, 2] developed stability and convergence results for stochastic parabolic partial differential equations in Hilbert spaces in the context of Lyapunov functional techniques. There is no previous research on stochastic equations with Markovian switching

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in infinite dimensions. This paper contributes to the existing literature by investigating the existence, uniqueness results and Feller property of stochastic heat equations with Markovian switching (SHEs-MS in short). We consider the heat conduction over thin wire $x \in D$ with a constant thermal diffusivity $\alpha > 0, k > 0$. Let $X(x,t)$ denote the temperature distribution at point x and time $t > 0$ due to a spatially dependent white noise. Let $\theta(t)$ denote a continuous-time Markov chain taking values on a finite state space $\mathbb{S} = \{1,2,\ldots N\}$. Suppose both ends of the wire are maintained at the freezing temperature. Then, given an initial temperature $h \in H = L^2(D)$, the temperature field $X(x,t)$ is governed by the initial-boundary value problem for SHEs-MS:

$$\begin{cases} \dfrac{\partial X(x,t)}{\partial t} = (\kappa\Delta - \alpha)X(x,t) + f\big(X(x,t),t,\theta(t)\big) + \sigma\big(X(x,t),t,\theta(t)\big)\dot{W}(x,t); \\ X|_{\partial D} = 0; \\ X(x,0) = h(x) \in H; \theta(0) = i \in \mathbb{S}, \qquad x \in D, t \in (0,T); \end{cases} \tag{1.1}$$

Here the state vector has two components $X(\cdot,t)$ and $\theta(t)$, the first one is in general referred to as the state while the second is regarded as the mode. In its operation, the system will switch from one mode to another in a random way, and the switching between the modes is governed by a Markov chain.

The purpose of this paper is to devote SHEs-MS (1.1). In the first part, the equation (1.1) is treated as nonlinear stochastic evolution equations in $L^2(D)$. The existence and uniqueness of mild solutions is verified through the semigroup approach, which is treated extensively in the book by Da Prato and Zabczyk [4]. In the second part, we show the pair of process $(X(t),\theta(t))$ being a Markov process with initial data $(h,i) \in H \times \mathbb{S}$, then the Feller property will be obtained in the similar way as Mao and Yuan [11].

The organization of this paper is as follows: we state basic definitions in section 2. We study the existence and uniqueness of solutions of equation (1.1) in section 3, and investigate the Feller property of the solution in section 4. Then we demonstrate an example in section 5.

2. PRELIMINARIES

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Without loss of generality, assume the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual conditions (i.e. it is right continuous with $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets). Let $\theta(t), t \geq 0$ be a right-continuous Markov chain on probability space taking values in a finite space $\mathbb{S} = \{1,2,\ldots N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{\theta(t+\delta) = j \mid \theta(t) = i\} = \begin{cases} \gamma_{ij}\delta + o(\delta), if i \neq j, \\ 1 + \gamma_{ii}\delta + o(\delta), if i = j, \end{cases}$$

for $\delta > 0$. Here $\gamma_{ij} > 0$, it is the transition rate from i to j, if $i \neq j$ while $\gamma_{ii} = -\sum_{i\neq j}\gamma_{ij}$. Let $D \subset \mathbb{R}^d$ be a bounded domain with a smooth boundary ∂D. Denote $L^2(D) = H = K$ with inner products $<\cdot,\cdot>_H, <\cdot,\cdot>_K$ and norms $\|\cdot\|_H, \|\cdot\|_K$, respectively. Denote by $\mathcal{L}(K,H)$ be the space of bounded linear operators from K into H, equipped with the usual operator norm $\|\cdot\|$. We introduce the subspace $K_0 = Q^{\frac{1}{2}}(K)$ of K which, endowed with the inner product

$$< u, v >_{K_0} = < Q^{-\frac{1}{2}}u, Q^{-\frac{1}{2}}v >_K,$$

is a Hilbert space, where Q is positive, self-adjoint, trace class operator on K. Let $\mathcal{L}_2^0 = \mathcal{L}_2(K_0, H)$ denote the space of all Hilbert-Schmidt operators from K_0 into H. The space $\mathcal{L}_2^0$ is also a separable Hilbert space, equipped with the norm

$$\|\phi\|_{\mathcal{L}_2^0}^2 = tr\left(\left(\phi Q^{\frac{1}{2}}\right)\left(\phi Q^{\frac{1}{2}}\right)^*\right), for\ any\ \phi \in \mathcal{L}_2^0 .$$

Clearly, for any bounded operators $\phi \in \mathcal{L}(K,H)$, this norm reduces to $\|\phi\|_{\mathcal{L}_2^0}^2 = tr(\phi Q \phi^*)$.

In this paper, we consider the following system of SHEs-MS:

$$\frac{\partial X(x,t)}{\partial t} = (\kappa\Delta - \alpha)X(x,t) + f\big(X(x,t),t,\theta(t)\big) + \sigma\big(X(x,t),t,\theta(t)\big)\dot{W}(x,t);$$

$$X|_{\partial D} = 0;$$

$$X(x,0) = h(x) \in H;\ \theta(0) = i \in \mathbb{S}; \qquad x \in D, t \in (0,T); \qquad (2.1)$$

where $\Delta = \sum_{i=1}^{d}\frac{\partial^2}{\partial x^2}$ is the Laplacian operator; κ, α are positive constants; $\theta(t)$ be the right-continuous Markov chain starting from the finite state $i \in \mathbb{S}$ at $t = 0$; the non-linear terms $f(u,t,r)$ and $\sigma(u,t,r)$ satisfy properly formulated Lipschitz and linear growth conditions; $\dot{W}(x,t)$be a spatially dependent white noise, by convention, is the formal time derivative $\frac{\partial}{\partial t}W(x,t)$ of the Wiener random field $W(x,t)$; let $W(\cdot,t)$ be an K-valued Wiener process with mean zero and covariance operator $Q, trQ < \infty$; initial value $h(x)$ is a given function in $H = L^2(D)$, and initial value $i \in \mathbb{S}$, both of them is $\mathcal{F}_0$-measurable.

3. THE EXISTENCE AND UNIQUNESS OF MILD SOLUTIONS

In this paper, we consider the SHEs-MS (2.1) as the following stochastic evolution equation

$$dX(t) = \big[AX(t) + F\big(X(t),t,\theta(t)\big)\big]dt + \Sigma\big(X(t),t,\theta(t)\big)dW(t), \qquad t \geq 0,$$

$$X(0) = h \in H, \theta(0) = i \in \mathbb{S}. \qquad (3.1)$$

By convention, we write $X(t) = X(\cdot,t)$, $F\big(X(t),t,\theta(t)\big) = f\big(X(\cdot,t),t,\theta(t)\big)$, $\Sigma\big(X(t),t,\theta(t)\big) = \sigma\big(X(\cdot,t),t,\theta(t)\big)$ and $dW(t) = W(\cdot,dt)$. Here $A = (\kappa\Delta - \alpha)$ for $\kappa > 0, \alpha > 0$; $A: D(A) \subset H \to H$ generates a contraction semigroup $G(t), t \geq 0$, on $H = L^2(D)$, which satisfies the properties: $\|G(t)h\| \leq \|h\|$, for $h \in H$ and $(A\varphi, \varphi) \leq 0$ for $\varphi \in D(A)$; $W(t)$ be a Q-Wiener process in $K = L^2(D)$ with $trQ < \infty$; initial values $h \in H = L^2(D)$, and $\theta(0) = i \in \mathbb{S}$ are $\mathcal{F}_0$-measurable. Throughout this chapter, for the existence and uniqueness of the mild solutions, we shall impose the following assumptions:

Hypothesis 3.1.

(H1) ***(Lipschitz condition)*** *$F(u,t,r)$ and $\Sigma(u,t,r)$ are predictable random fields. There exists a constant $K_1 > 0$, such that*

$$\|F(u,t,r) - F(v,t,r)\|_H^2 + \| \Sigma(u,t,r) - \Sigma(v,t,r) \|_{\mathcal{L}_2^0}^2 \leq K_1 \|u - v\|_H^2,$$

for any $u, v \in H, t \in [0,T], r \in \mathbb{S}$.

(H2) ***(Linear growth condition)*** *There exists a constant $K_2 > 0$, such that*

$$\|F(u,t,r)\|_H^2 + \| \Sigma(u,t,r) \|_{\mathcal{L}_2^0}^2 \leq K_2(1 + \|u\|_H^2),$$

for any $(u,t,r) \in H \times [0,T] \times \mathbb{S}$.

Definition 3.2. *A H-valued stochastic process $X(t), t \in [0,T]$, is called a mild solution of (3.1) if*

(i) $X(t)$ *is adapted to $\mathcal{F}_t$ and continuous in t;*

(ii) $X(t)$ *is measurable and $\int_0^T \|X(t)\|_H^2 dt < \infty$;*

(iii) *For any $t \in [0,T]$, equation*

$$X(t) = G(t)h + \int_0^t G(t-s)F\big(X(s), s, \theta(s)\big)ds + \int_0^t G(t-s)\Sigma\big(X(s), s, \theta(s)\big)dW(s). \quad (3.2)$$

Theorem 3.3 ([3]). *Suppose that $G(t)$ be a contraction semigroup on H and for arbitrary $\mathcal{L}_2^0$-valued predictable process $\phi(t), t \in [0,T]$ satisfies the condition:*

$$E\int_0^T [tr(\phi(s)Q\phi^*(s)]^p \, ds < \infty,$$

for $p \geq 1$. Then there exists constant $C_p > 0$ such that, for any $t \in [0,T]$,

$$E sup_{0 \leq r \leq t} \| \int_0^r G(t-s)\phi(s)dW(s) \|_H^{2p} \leq C_p E\left[\int_0^t \| \phi(s) \|_{\mathcal{L}_2^0}^2 ds\right]^p,$$

where $\| \phi(s) \|_{\mathcal{L}_2^0}^2 = tr\big(\phi(s)Q\phi^*(s)\big)$. □

The following theorem shows that the existence and uniqueness of mild solutions of SHEs-MS (3.1).

Theorem 3.4 (**Existence and Uniqueness Theorem**). *Suppose that the conditions (H1) and (H2) are satisfied, and let* h *be a* $\mathcal{F}_0$*-measurable such that* $E \parallel h \parallel^{2p} < \infty$, *for* $p \geq 1$. *Then SHEs-MS (3.1) has a unique (mild) solution* $X(t), t \in [0,T]$ *which is a continuous adapted process in* H *such that* $X \in L^{2p}\big(\Omega; C([0,T]; H)\big)$ *satisfying*

$$E sup_{0 \leq t \leq T} \parallel X(t) \parallel^{2p} \leq b_2\{1 + E \parallel h \parallel^{2p}\}, \qquad (3.3)$$

for some constant $b_2 > 0,\ depending\ on\ T, p, K_2.$

Proof: **Step 1**. **Uniqueness.** Suppose that $X(t)$ and $Y(t)$ are both mild solutions satisfying the integral equation (3.2). Let $C = sup_{0 \leq t \leq T} \|G(t)\|_{\mathcal{L}(H)}$. By making use of theorem 3.3, the hölder inequality and Lipschitz condition (H1), we consider

$$E\|X(t) - Y(t)\|_H^2 = E\left\|\int_0^t G(t-s)\big[F\big(X(s), s, \theta(s)\big) - F\big(Y(s), s, \theta(s)\big)\big] ds\right.$$

$$\left. + \int_0^t G(t-s)\big[\Sigma\big(X(s), s, \theta(s)\big) - \Sigma\big(Y(s), s, \theta(s)\big)\big] dW(s)\right\|_H^2$$

$$\leq 2\{C^2 E \left\|\int_0^t \big[F\big(X(s), s, \theta(s)\big) - F\big(Y(s), s, \theta(s)\big)\big] ds\right\|_H^2$$

$$+4E\int_0^t \big\|\Sigma\big(X(s), s, \theta(s)\big) - \Sigma\big(Y(s), s, \theta(s)\big)\big\|_{\mathcal{L}_2^0}^2 ds\}$$

$$\leq 2\{C^2 T E \int_0^t \big\|F\big(X(s), s, \theta(s)\big) - F\big(Y(s), s, \theta(s)\big)\big\|_H^2 ds$$

$$+4E\int_0^t \big\|\Sigma\big(X(s), s, \theta(s)\big) - \Sigma\big(Y(s), s, \theta(s)\big)\big\|_{\mathcal{L}_2^0}^2 ds\}$$

$$\leq 2K_1(C^2 T + 4)\int_0^t E\|X(s) - Y(s)\|_H^2 ds.$$

It then follows from the Gronwall inequality that

$$E\|X(t) - Y(t)\|_H^2 = 0,$$

for all t in $[0, T]$. Therefore the solution is unique.

Step 2. Existence. This proof is based on the standard contraction mapping principle.

Let $\mathcal{H}_{p,T}, p \geq 1$ denote the set of all continuous $\mathcal{F}_t$-adapted processes in H for $0 \leq t \leq T$ such that

$$Esup_{0\leq t\leq T} \parallel X(t) \parallel^{2p} < \infty, for\ a\ given\ p \geq 1. \qquad (3.4)$$

Then $\mathcal{H}_{p,T}$ is a Banach space under the norm:

$$\parallel X \parallel_{p,T} = \{Esup_{0\leq t\leq T} \parallel X(t) \parallel^{2p}\}^{1/2p}. \qquad (3.5)$$

Define an operator $\Lambda : \mathcal{H}_{p,T} \to \mathcal{H}_{p,T}$ as follows:

$$\Lambda(t)X = G(t)h + \int_0^t G(t-s)F\big(X(s), s, \theta(s)\big)ds$$

$$+ \int_0^t G(t-s)\Sigma\big(X(s), s, \theta(s)\big)dW(s), \qquad (3.6)$$

for $t \in [0,T], X \in \mathcal{H}_{p,T}$.

To show the operator Λ is well defined, let $I_1(t) = G(t)h$;

$$I_2(t) = \int_0^t G(t-s)F(X(s), s, \theta(s))ds\,; \;\; I_3(t) = \int_0^t G(t-s)\Sigma\big(X(s), s, \theta(s)\big)dW(s).$$

So that $\Lambda(t)X = I_1(t) + I_2(t) + I_3(t)$. Then

$$\|\Lambda X\|_{p,T} = \{Esup_{0\leq t\leq T} \parallel \Lambda(t)X \parallel^{2p}\}^{1/2p}$$

$$= \{Esup_{0\leq t\leq T} \parallel I_1(t) + I_2(t) + I_3(t) \parallel^{2p}\}^{1/2p}. \quad (3.7)$$

We shall estimate (3.7) separately. First, we have

$$Esup_{0\leq t\leq T} \parallel I_1(t) \parallel^{2p} = Esup_{0\leq t\leq T} \parallel G(t)h \parallel^{2p} \leq C^{2p}E\|h\|^{2p} \leq C_1E\|h\|^{2p}. \quad (3.8)$$

By making use of the Hölder inequality and the linear growth condition (H2), we have

$$Esup_{0\leq t\leq T} \parallel I_2(t) \parallel^{2p} = Esup_{0\leq t\leq T} \parallel \int_0^t G(t-s)F\left(X(s), s, \theta(s)\right)ds \parallel^{2p}$$

$$\leq C^{2p}E \parallel \int_0^T F\left(X(s), s, \theta(s)\right)ds \parallel^{2p}$$

$$\leq C^{2p}T^{2p-1}E\int_0^T \parallel F\left(X(s), s, \theta(s)\right) \parallel^{2p} ds$$

$$\leq C^{2p}T^{2p-1}K_2^pE\int_0^T (1+\| X(s) \|^2)^p ds$$

$$\leq C^{2p}T^{2p-1}K_2^p 2^{p-1}E\int_0^T (1+\| X(s) \|^{2p}) ds$$

$$\leq C^{2p}T^{2p}K_2^p 2^{p-1}[1+\| X \|_{p,T}^{2p}]$$

$$\leq C_2(1+\| X \|_{p,T}^{2p}), \qquad (3.9)$$

where the positive constant C_2, depending on p, T, K_2.

Similarly, by theorem 3.3, the Hölder inequality and the linear growth condition (H2), we have

$$Esup_{0\leq t\leq T} \| I_3(t) \|^{2p} = Esup_{0\leq t\leq T} \| \int_0^t G(t-s)\Sigma\big(X(s), s, \theta(s)\big)dW(s) \|^{2p}$$

$$\leq C_pE\left[\int_0^T \| \Sigma\big(X(s), s, \theta(s)\big) \|_{\mathcal{L}_2^0}^2 ds\right]^p$$

$$\leq K_2^pT^{p-1}C_pE\int_0^T (1+\| X(s) \|^2)^p ds$$

$$\leq K_2^pT^{p-1}C_p2^{p-1}E\int_0^T (1+\| X(s) \|^{2p}) ds$$

$$\leq K_2^p2^{p-1}T^pC_p(1+\| X \|_{p,T}^{2p})$$

$$\leq C_3\big(1+\| X \|_{p,T}^{2p}\big), \qquad (3.10)$$

where the positive constant C_3, depending on p, T, K_2.

Therefore, substitute (3.8) (3.9) (3.10) into (3.7), we have

$$\|\Lambda X\|_{p,T}^{2p} = Esup_{0\leq t\leq T} \| \Lambda(t)X \|^{2p}$$

$$\leq 3^{2p-1}\{C_1E \| h \|^{2p} + C_2\big(1+\| X \|_{p,T}^{2p}\big) + C_3\big(1+\| X \|_{p,T}^{2p}\big)\}$$

$$\leq b_1\big\{1+E \| h \|^{2p} + \| X \|_{p,T}^{2p}\big\},$$

where $b_1 > 0$ is a constant depending on T, K_2, p. It is shown that the map $\Lambda: \mathcal{H}_{p,T} \to \mathcal{H}_{p,T}$ is well-defined as asserted.

Step 3. We will show that Λ is a contraction mapping for a small $T = T_1$. To this end, for $X, X' \in \mathcal{H}_{p,T}$, we define

$$J_1(t) = \int_0^t G(t-s)\big[F\big(X(s), s, \theta(s)\big) - F\big(X'(s), s, \theta(s)\big)\big]ds,$$

$$J_2(t) = \int_0^t G(t-s)\big[\Sigma\big(X(s), s, \theta(s)\big) - \Sigma\big(X'(s), s, \theta(s)\big)\big]dW(s).$$

Then

$$\|\Lambda X - \Lambda X'\|_{p,T}^{2p} = Esup_{0\le t\le T}\|\Lambda X(t) - \Lambda X'(t)\|^{2p}$$

$$= Esup_{0\le t\le T}\left\|\int_0^t G(t-s)\big[F\big(X(s), s, \theta(s)\big) - F\big(X'(s), s, \theta(s)\big)\big]ds\right.$$

$$\left.+\int_0^t G(t-s)\big[\Sigma\big(X(s), s, \theta(s)\big) - \Sigma\big(X'(s), s, \theta(s)\big)\big]dW(s)\right\|^{2p}$$

$$= Esup_{0\le t\le T}\|J_1(t) + J_2(t)\|^{2p}$$

$$\le 2^{2p-1}(\,\|J_1\|_{p,T}^{2p} + \|J_2\|_{p,T}^{2p}). \qquad (3.11)$$

Then, using the Hölder inequality, and the Lipschitz condition (H1), we get

$$\|J_1\|_{p,T}^{2p} = Esup_{0\le t\le T}\left\|\int_0^t G(t-s)\big[F\big(X(s), s, \theta(s)\big) - F\big(X'(s), s, \theta(s)\big)\big]ds\right\|^{2p}$$

$$\le C^{2p}E\left\|\int_0^T \big[F\big(X(s), s, \theta(s)\big) - F\big(X'(s), s, \theta(s)\big)\big]ds\right\|^{2p}$$

$$\le C^{2p}T^{2p-1}E\int_0^T \big\|F\big(X(s), s, \theta(s)\big) - F\big(X'(s), s, \theta(s)\big)\big\|^{2p}ds$$

$$\le C^{2p}T^{2p-1}K_1^pE\int_0^T \|X(s) - X'(s)\|^{2p}\,ds$$

$$\le C^{2p}T^{2p}K_1^p\,\|X - X'\|_{p,T}^{2p}, \qquad (3.12)$$

and using the Hölder inequality, the Lipschitz condition (H1) and theorem 3.3, we have

$$\|J_2\|_{p,T}^{2p} = Esup_{0\le t\le T}\left\|\int_0^t G(t-s)\big[\Sigma\big(X(s), s, \theta(s)\big) - \Sigma\big(X'(s), s, \theta(s)\big)\big]dW(s)\right\|^{2p}$$

$$\leq C_p E\left[\int_0^T \| \Sigma\big(X(s),s,\theta(s)\big) - \Sigma\big(X'(s),s,\theta(s)\big) \|_{\mathcal{L}_2^0}^2 \, ds\right]^p$$

$$\leq C_p T^{p-1} E \int_0^T \| \Sigma\big(X(s),s,\theta(s)\big) - \Sigma\big(X'(s),s,\theta(s)\big) \|_{\mathcal{L}_2^0}^{2p} \, ds$$

$$\leq C_p T^{p-1} K_1^p E \int_0^T \| X(s) - X'(s) \|^{2p} \, ds$$

$$\leq C_p T^p K_1^p \| X - X' \|_{p,T}^{2p}. \qquad (3.13)$$

Therefore, substitute (3.12) and (3.13) into (3.11), we have

$$\|\Lambda X - \Lambda X'\|_{p,T}^{2p} = E sup_{0\leq t\leq T} \| \Lambda(t)X - \Lambda(t)X' \|^{2p}$$

$$\leq 2^{2p-1}(\| J_1\|_{p,T}^{2p} + \| J_2\|_{p,T}^{2p})$$

$$\leq 2^{2p-1}\big(C^{2p}T^{2p}K_1^p + C_p T^p K_1^p\big) \| X - X' \|_{p,T}^{2p}$$

$$\leq 2^{2p} K_1^p \big(C^{2p} + C_p\big) T^{2p} \| X - X' \|_{p,T}^{2p}.$$

Or $\quad \|\Lambda X - \Lambda X'\|_{p,T} \leq \rho(T)\|X - X'\|_{p,T}$,

where $\rho(T) = 2\sqrt{K_1}(C^{2p} + C_p)^{1/2p}T$. Let $T = T_1$ be sufficiently small so that $\rho(T_1) < 1$. Then Λ is a Lipschitz-continuous contraction mapping in $\mathcal{H}_{p,T}$ which has a unique fixed point X. This element $X(t)$ is the unique mild solution of (3.1). For $T > T_1$, we can extend the solution by continuation, from T_1 to T_2 and so on. This completes the existence and uniqueness proof.

Step 4. We verify the inequality (3.3). We use the linear growth condition (H2), theorem 3.3, and the Hölder inequality,

$$E sup_{0\leq t\leq T} \| X(t) \|^{2p} \leq 3^{2p-1}\{E sup_{0\leq t\leq T} \| G(t)h \|^{2p}$$

$$+E sup_{0\leq t\leq T} \| \int_0^t G(t-s)F\big(X(s),s,\theta(s)\big)ds \|^{2p}$$

$$+E sup_{0\leq t\leq T} \| \int_0^t G(t-s)\Sigma\big(X(s),s,\theta(s)\big)dW(s) \|^{2p}\}$$

$$\leq 3^{2p-1}\{C^{2p} E \| h \|^{2p} + C^{2p} E[\| \int_0^T F\big(X(s),s,\theta(s)\big)\, ds \|^2]^p$$

$$+C_pE\left[\int_0^T \| \Sigma\big(X(s),s,\theta(s)\big) \|_{\mathcal{L}_2^0}^2 ds\right]^p \}$$

$$\leq 3^{2p-1}\{C^{2p}E \| h \|^{2p} + C^{2p}T^pE[\int_0^T \|F\big(X(s),s,\theta(s)\big)\|^2 ds]^p$$

$$+C_pE[\int_0^T \| \Sigma\big(X(s),s,\theta(s)\big) \|_{\mathcal{L}_2^0}^2 ds]^p\}$$

$$\leq 3^{2p-1}\{C^{2p}E \| h \|^{2p} + C^{2p}T^pK_2^pE[\int_0^T (1+\|X(s)\|^2)\, ds]^p$$

$$+C_pK_2^pE[\int_0^T (1+\|X(s)\|^2)\, ds]^p\}$$

$$\leq 3^{2p-1}\{C^{2p}E \| h \|^{2p} + (C^{2p}T^pK_2^p + C_pK_2^p)E[\int_0^T (1+\|X(s)\|^2)\, ds]^p\}$$

$$\leq 3^{2p-1}\{C^{2p}E \| h \|^{2p} + 2^{p-1}K_2^p(C^{2p}T^p + C_p)[T^p + E[\int_0^T \|X(s)\|^2 ds]^p]\}$$

$$\leq 3^{2p-1}\big\{C^{2p}E \| h \|^{2p} + 2^{p-1}K_2^p\big(C^{2p}T^p + C_p\big)[T^p + T^{p-1}E\int_0^T \|X(s)\|^{2p}ds]\}$$

$$\leq 3^{2p-1}\big\{C^{2p}E \| h \|^{2p} + 2^{p-1}K_2^p\big(C^{2p}T^p + C_p\big)T^p\left[1 + T^{\frac{p-1}{p}}E\int_0^T \|X(s)\|^{2p}ds\right]\}$$

$$\leq C_4\left\{1 + E \| h \|^{2p} + T^{\frac{p-1}{p}}\int_0^T Esup_{0\leq s\leq t} \| X(s) \|^{2p} dt\right\},$$

which, by the Gronwall inequality, implies the inequality (3.3). The constant b_2 depends on T, p, K_2. □

4. FELLER PROPERTY

In this section, we study $\big(X(t),\theta(t)\big)$ is a Markov process. To prove the following theorem, we can consider the theorem 3.27 in Mao and Yuan [11]. Denote by $\mathcal{B}_b(H)$ be the space of all real bounded and measurable functions in H. Denote by $C_b(H)$ be the set of all uniformly continuous and bounded functions in H, and by $C_b^k(H)$ be the set of all k times continuously differentiable functions with their first k derivatives bounded.

Theorem 4.1. *Let $X(t) \in H = L^2(D)$ be a solution of the equation*

$$dX(t) = \big[AX(t) + F\big(X(t),t,\theta(t)\big)\big]dt + \Sigma\big(X(t),t,\theta(t)\big)dW(t), t \geq 0; X(0) = h, \quad (4.1)$$

whose coefficients satisfy the conditions of the existence and uniqueness theorem 3.4. Then $\big(X(t),\theta(t)\big)$ *is a Markov process whose transition probability is defined by*

$$P(s,(h,i);t,A\times\{j\})=\mathbb{P}\left\{X_s^{h,i}(t)\in A\times\{j\}\right\}, \qquad (4.2)$$

for $(h,i)\in H\times\mathbb{S},\ A\in\mathcal{B}(H)$ *and* $j\in\mathbb{S}$, *where* $X_s^{h,i}(t)$ *be a solution of the problem (3.1) on* $t\ge s$ *with initial data* $X(s)=h\in H,\theta(s)=i\in\mathbb{S}$, *both of initial data is* $\mathcal{F}_s$*-measurable. That is*

$$X_s^{h,i}(t)=G(t-s)h+\int_s^t G(t-u)F\big(X_s^{h,i}(u),u,\theta_s^i(u)\big)du$$

$$+\int_s^t G(t-u)\Sigma\left(X_s^{h,i}(u),u,\theta_s^i(u)\right)dW(u), \qquad (4.3)$$

on $t\ge s$, *where* $\theta_s^i(t)$ *stands for the Markov chain on* $t\ge s$ *starting from state* i *at time* $t=s$. □

Next, we consider the Feller property of the Markov process $(X(t),\theta(t))$ in $H\times\mathbb{S}$.

Lemma 4.2. *Assume that hypothesis 3.1 holds. For* $(h,i)\in H\times\mathbb{S}$, *and* $0\le s\le t<\infty$, *let* $X_s^{h,i}(t)$ *and* $\theta_s^i(t)$ *are defined in theorem 4.1. Then for any* $T>0$,

$$E\left\|X_s^{h,i}(t)-X_u^{q,i}(t)\right\|_H^2\le\{5C\,\|G(u-s)h-q\|^2+[5CK_2(u-s)^2+20K_2(u-s)$$
$$+(40CK_2T^2+160K_2T)\big(C'(u-s)+o(u-s)\big)][1$$
$$+b_2(1+E\|h\|^2)]\}\exp(10CT^2K_1+40K_1T),$$

if $0\le s<u<t\le T$, h *is* $\mathcal{F}_s$*-measurable,* q *is* $\mathcal{F}_u$*-measurable and* $E\parallel h\parallel^2<\infty$ *and* $E\parallel q\parallel^2<\infty$ *where* C, C' *are positive constants.*

Proof: For $0\le s\le u\le t\le T$, $p\ge 1$, we get

$E\left\|X_s^{h,i}(t)-X_u^{q,i}(t)\right\|_H^2$

$$=E\left\|G(t-s)h-G(t-u)q+\int_s^t G(t-v)F\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)dv\right.$$

$$-\int_u^t G(t-v)F\left(X_u^{q,i}(v),v,\theta_u^i(v)\right)dv+\int_s^t G(t-v)\Sigma\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)dW(v)$$

$$-\int_u^t G(t-v)\Sigma\left(X_u^{q,i}(v),v,\theta_u^i(v)\right)dW(v)\Bigg\|_H^2$$

$$=E\Bigg\|G(t-s)h-G(t-u)q+\int_s^t G(t-v)F\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)dv$$

$$-\int_u^t G(t-v)\,F\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)dv$$

$$+\int_u^t G(t-v)\left[F\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)-F\left(X_u^{q,i}(v),v,\theta_u^i(v)\right)\right]dv$$

$$+\int_s^t G(t-v)\Sigma\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)dW(v)$$

$$-\int_u^t G(t-v)\Sigma\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)dW(v)$$

$$+\int_u^t G(t-v)\left[\Sigma\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)-\Sigma\left(X_u^{q,i}(v),v,\theta_u^i(v)\right)\right]dW(v)\Bigg\|_H^2$$

$$=E\Bigg\|G(t-s)h-G(t-u)q+\int_s^u G(t-v)F\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)dv$$

$$+\int_u^t G(t-v)\left[F\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)-F\left(X_u^{q,i}(v),v,\theta_u^i(v)\right)\right]dv$$

$$+\int_s^u G(t-v)\,\Sigma\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)dW(v)$$

$$+\int_u^t G(t-v)\left[\Sigma\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)-\Sigma\left(X_u^{q,i}(v),v,\theta_u^i(v)\right)\right]dW(v)\Bigg\|_H^2$$

$$\leq 5\{E\|G(t-s)h-G(t-u)q\|_H^2+E\left\|\int_s^u G(t-v)F\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)dv\right\|_H^2$$

$$+E\left\|\int_s^u G(t-v)\Sigma\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)dW(v)\right\|_H^2$$

$$+E\left\|\int_u^t G(t-v)\left[F\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)-F\left(X_u^{q,i}(v),v,\theta_u^i(v)\right)\right]dv\right\|_H^2$$

$$+E\left\|\int_u^t G(t-v)\left[\Sigma\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)-\Sigma\left(X_u^{q,i}(v),v,\theta_u^i(v)\right)\right]dW(v)\right\|_H^2\}. \quad (4.4)$$

Using the Hölder inequality, the linear growth condition (H2), and theorem 3.4, then there exists a positive constant b_2,

$$E\left\|\int_s^u G(t-v)F\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)dv\right\|_H^2 \leq C(u-s)E\int_s^u \left\|F\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)\right\|_H^2 dv$$

$$\leq C(u-s)K_2E\int_s^u\left(1+\left\|X_s^{h,i}(v)\right\|_H^2\right)dv$$

$$\leq C(u-s)^2K_2\left(1+b_2(1+E\|h\|^2)\right), \quad (4.5)$$

where $\| G(t) \|_{\mathcal{L}(H)}^2 \leq C$.

Using theorem 3.3, the linear growth condition (H2), and theorem 3.4, then there exists a positive constant b_2,

$$E\left\|\int_s^u G(t-v)\Sigma\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)dW(v)\right\|_H^2 \leq 4E\int_s^u \|\Sigma\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)\|_{\mathcal{L}_2^0}^2 dv$$

$$\leq 4K_2E\int_s^u\left(1+\left\|X_s^{h,i}(v)\right\|_H^2\right)dv$$

$$\leq 4K_2(u-s)\left(1+b_2(1+E\|h\|^2)\right). \quad (4.6)$$

Using the Hölder inequality and the Lipschitz condition (H1), we have

$$E\left\|\int_u^t G(t-v)\left[F\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)-F\left(X_u^{q,i}(v),v,\theta_u^i(v)\right)\right]dv\right\|_H^2$$

$$\leq C(t-u)E\int_u^t\left\|F\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)-F\left(X_u^{q,i}(v),v,\theta_u^i(v)\right)\right\|_H^2 dv$$

$$\leq 2C(t-u)\{E\int_u^t\left\|F\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)-F\left(X_u^{q,i}(v),v,\theta_s^i(v)\right)\right\|_H^2 dv$$

$$+E\int_u^t\left\|F\left(X_u^{q,i}(v),v,\theta_s^i(v)\right)-F\left(X_u^{q,i}(v),v,\theta_u^i(v)\right)\right\|_H^2 dv\}$$

$$\leq 2CT\left\{K_1\int_u^t E\left\|X_s^{h,i}(v)-X_u^{q,i}(v)\right\|_H^2 dv\right.$$

$$\left.+E\int_u^t \left\|F\left(X_u^{q,i}(v),v,\theta_s^i(v)\right)-F\left(X_u^{q,i}(v),v,\theta_u^i(v)\right)\right\|_H^2 dv\right\}. \quad (4.7)$$

Note that $X_u^{q,i}(v)$ and the indicator function $I_{\{\theta_s^i(v)\neq\theta_u^i(v)\}}$ are conditional independent with respect to the σ $-$algebra generated by $\theta_s^i(v)$. We compute, using the linear growth condition (H2), that

$$E\int_u^t \left\|F\left(X_u^{q,i}(v),v,\theta_s^i(v)\right)-F\left(X_u^{q,i}(v),v,\theta_u^i(v)\right)\right\|_H^2 dv$$

$$\leq 2E\int_u^t [\left\|F\left(X_u^{q,i}(v),v,\theta_s^i(v)\right)\right\|_H^2+\left\|F\left(X_u^{q,i}(v),v,\theta_u^i(v)\right)\right\|_H^2] I_{\{\theta_s^i(v)\neq\theta_u^i(v)\}}dv$$

$$\leq 4K_2 E\int_u^t (1+\left\|X_u^{q,i}(v)\right\|_H^2)\, I_{\{\theta_s^i(v)\neq\theta_u^i(v)\}}dv$$

$$\leq 4K_2\int_u^t E[E(1+\left\|X_u^{q,i}(v)\right\|_H^2) I_{\{\theta_s^i(v)\neq\theta_u^i(v)\}}|\theta_s^i(v)]\, dv$$

$$\leq 4K_2\int_u^t E\left[E\left[\left(1+\left\|X_u^{q,i}(v)\right\|_H^2\right)\middle|\theta_s^i(v)\right]E\left[I_{\{\theta_s^i(v)\neq\theta_u^i(v)\}}\middle|\theta_s^i(v)\right]\right]dv,$$

But, by the Markov property,

$$E\left[I_{\{\theta_s^i(v)\neq\theta_u^i(v)\}}\middle|\theta_s^i(v)\right]=\sum_{j\in\mathbb{S}} I_{\{\theta_s^i(v)=j\}}\mathbb{P}\{\theta_u^i(v)\neq j|\theta_s^i(v)=j\}$$

$$=\sum_{j\in\mathbb{S}} I_{\{\theta_s^i(v)=j\}}\sum_{j\neq k}(\gamma_{jk}(u-s)+o(u-s))$$

$$\leq \max_{i\in\mathbb{S}}(-\gamma_{ii})\,(u-s)+o(u-s)$$

$$\leq C'(u-s)+o(u-s), \quad (4.8)$$

then, applying theorem 3.4, there exists a positive constant b_2,

$$E\int_u^t \left\|F\left(X_u^{q,i}(v),v,\theta_s^i(v)\right)-F\left(X_u^{q,i}(v),v,\theta_u^i(v)\right)\right\|_H^2 dv$$

$$\leq 4K_2T[1+b_2(1+E\|h\|^2)][C'(u-s)+o(u-s)]. \quad (4.9)$$

Hence (4.7) becomes

$$E\left\|\int_u^t G(t-v)\left[F\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)-F\left(X_u^{q,i}(v),v,\theta_u^i(v)\right)\right]dv\right\|_H^2$$

$$\leq 2CT\left\{4K_2T[1+b_2(1+E\|h\|^2)][C'(u-s)+o(u-s)]\right.$$

$$\left.+K_1\int_u^t E\left\|X_s^{h,i}(v)-X_u^{q,i}(v)\right\|_H^2dv\right\}. \qquad (4.10)$$

Then, using theorem 3.3 we can show

$$E\left\|\int_u^t G(t-v)\left[\Sigma\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)-\Sigma\left(X_u^{q,i}(v),v,\theta_u^i(v)\right)\right]dW(v)\right\|_H^2$$

$$\leq 4E\int_u^t \|\Sigma\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)-\Sigma\left(X_u^{q,i}(v),v,\theta_u^i(v)\right)\|_{\mathcal{L}_2^0}^2dv$$

$$\leq 8\left\{E\int_u^t \left\|\Sigma\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)-\Sigma\left(X_u^{q,i}(v),v,\theta_s^i(v)\right)\right\|_{\mathcal{L}_2^0}^2dv\right.$$

$$\left.+E\int_u^t \left\|\Sigma\left(X_u^{q,i}(v),v,\theta_s^i(v)\right)-\Sigma\left(X_u^{q,i}(v),v,\theta_u^i(v)\right)\right\|_{\mathcal{L}_2^0}^2dv\right\}, \qquad (4.11)$$

Using the Lipschitz condition (H1), we have

$$E\int_u^t \|\Sigma\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)-\Sigma\left(X_u^{q,i}(v),v,\theta_s^i(v)\right)\|_{\mathcal{L}_2^0}^2dv$$

$$\leq K_1\int_u^t E\left\|X_s^{h,i}(v)-X_u^{q,i}(v)\right\|_H^2dv. \qquad (4.12)$$

We use the similar way as (4.8)-(4.9) to get

$$E\int_u^t \|\Sigma\left(X_u^{q,i}(v),v,\theta_s^i(v)\right)-\Sigma\left(X_u^{q,i}(v),v,\theta_u^i(v)\right)\|_{\mathcal{L}_2^0}^2dv$$

$$\leq 4K_2T[1+b_2(1+E\|h\|^2)][C'(u-s)+o(u-s)]. \qquad (4.13)$$

Hence, substitute (4.13) and (4.12) into (4.11), we have

$$E\left\|\int_u^t G(t-v)\left[\Sigma\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)-\Sigma\left(X_u^{q,i}(v),v,\theta_u^i(v)\right)\right]dW(v)\right\|_H^2$$

$$\leq 8\{4K_2T[1+b_2(1+E\|h\|^2)][C'(u-s)+o(u-s)]$$

$$+K_1\int_u^t E\left\|X_s^{h,i}(v)-X_u^{q,i}(v)\right\|_H^2dv\}. \qquad (4.14)$$

Using the properties of the semigroup of bounded linear operators on H, we have

$$\begin{aligned}\|G(t-s)h-G(t-u)q\|^2 &= \|G(t-u+u-s)h-G(t-u)q\|^2 \\ &= \|G(t-u)(G(u-s)h)-G(t-u)q\|^2 \\ &= \|G(t-u)(G(u-s)h-q)\|^2 \\ &\leq C\|G(u-s)h-q\|^2. \qquad (4.15)\end{aligned}$$

Therefore, substitute (4.5) (4.6) (4.10) (4.14), and (4.15) into (4.4), we have

$$E\left\|X_s^{h,i}(t)-X_u^{q,i}(t)\right\|_H^2 \leq 5C\,\|(G(u-s)h-q)\|^2+[5CK_2(u-s)^2+20K_2(u-s)$$

$$+(40CK_2T^2+160K_2T)\big(C'(u-s)+o(u-s)\big)][1+b_2(1+E\|h\|^2)]$$

$$+(10CTK_1+40K_1)\left(\int_u^t E\left\|X_s^{h,i}(v)-X_u^{q,i}(v)\right\|_H^2 dv\right).$$

The required assertion finally follows from the Gronwall inequality. □

Theorem 4.3 (Feller Property). *The Markov process* $(X_s^{h,i}(t),\theta_s^i(t))$ *satisfies the Feller property. i.e. if for any* $i\in\mathbb{S},\delta>0$ *and any bounded continuous function* $\psi: H\to H$, *the mapping*

$$(h,s)\to\sum_{j\in\mathbb{S}}\int_{\mathbb{R}^d}\psi(y)\,P(s,(h,i);s+\delta,dy\times\{j\})=E\psi\left(X_s^{h,i}(s+\delta)\right),$$

is continuous.

Proof: For any bounded continuous function $\psi: H\to H$ and for any fixed $i\in\mathbb{S},\delta>0$. Note that for $s\leq u\leq t\leq T$,

$$\begin{aligned}&E\psi\left(X_s^{h,i}(s+\delta)\right)-E\psi\left(X_u^{q,i}(u+\delta)\right)\\ &\qquad=E\psi\left(X_s^{h,i}(s+\delta)\right)-E\psi\left(X_s^{h,i}(u+\delta)\right)+E\psi\left(X_s^{h,i}(u+\delta)\right)\\ &\qquad-E\psi\left(X_u^{q,i}(u+\delta)\right).\end{aligned}$$

By the lemma 4.2 and dominated convergence theorem

$$E\psi\left(X_s^{h,i}(u+\delta)\right)-E\psi\left(X_u^{q,i}(u+\delta)\right)\to 0,\qquad as\ (q,u)\to(h,s).$$

Then using the Hölder inequality, theorem 3.3 and the linear growth condition (H2) to consider

$$E\left\|X_s^{h,i}(t)-X_s^{h,i}(u)\right\|_H^2 = E\left\|G(t-s)h-G(u-s)h+\int_s^t G(t-v)F\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)dv\right.$$

$$-\int_s^u G(t-v)F\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)dv$$

$$+\int_s^t G(t-v)\Sigma\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)dW(v)$$

$$\left.-\int_s^u G(t-v)\Sigma\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)dW(v)\right\|_H^2$$

$$= E\left\|G(t-s)h-G(u-s)h+\int_u^t G(t-v)F\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)dv\right.$$

$$\left.+\int_u^t G(t-v)\Sigma\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)dW(v)\right\|_H^2$$

$$\leq 3\{E\|G(t-s)h-G(u-s)h\|_H^2$$

$$+\left\|E\int_u^t G(t-v)F\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)dv\right\|_H^2$$

$$+\left\|E\int_u^t G(t-v)\Sigma\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)dW(v)\right\|_H^2\}$$

$$\leq 3\{CE\|G(t-u)h-h\|_H^2$$

$$+C(t-u)E\int_u^t \left\|F\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)\right\|_H^2 dv$$

$$+4E\int_u^t \|\Sigma\left(X_s^{h,i}(v),v,\theta_s^i(v)\right)\|_{\mathcal{L}_2^0}^2 dv\}$$

$$\leq 3\left\{CE\|G(t-u)h-h\|_H^2\right.$$

$$\left.+[C(t-u)K_2+4K_2]E\int_u^t\left(1+\left\|X_s^{h,i}(v)\right\|_H^2\right)dv\right\}.$$

Then using theorem 3.4 to get

$$E\left\|X_s^{h,i}(t)-X_s^{h,i}(u)\right\|_H^2 \leq 3CE\|G(t-u)h-h\|_H^2+3K_2C(t-u)^2$$

$$+12K_2(t-u)\big(1+b_2(1+E\|h\|^2)\big).$$

From the above inequality, we obtain $X_s^{h,i}(s+\delta)$ convergences to $X_s^{h,i}(u+\delta)$ as $u \to s$.

Then using the dominated convergence theorem, we get

$$E\psi\left(X_s^{h,i}(s+\delta)\right)-E\psi\left(X_s^{h,i}(u+\delta)\right) \to 0, \quad as\ u \to s.$$

Therefore,

$$E\psi\left(X_s^{h,i}(s+\delta)\right) \to E\psi\left(X_u^{q,i}(u+\delta)\right), \ \ as\ (q,u) \to (h,s).$$

In other words, $E\psi\left(X_s^{h,i}(s+\delta)\right)$ as a function of (h,s) is a continuous and that is Feller property. □

5. EXAMPLE

In this section, we demonstrate an example about unique mild solution and Feller property of the following system of SHEs-MS (5.1).

Consider the following stochastic heat equations with Markovian switching

$$dy(x,t)=\frac{\partial^2}{\partial x^2}y(x,t)dt+\sigma f\big(y(x,t),\theta(t)\big)dW(t), t \geq 0, 0<x<1,$$

$$y(0,t)=y(1,t)=0, t \geq 0; y(x,0)=y_0(x); 0 \leq x \leq 1;\ \theta(0)=\theta_0 \in \mathbb{S}, \quad (5.1)$$

where $W(t), t \geq 0,$ is a real standard Brownian motion, $\theta(t), t \geq 0,$ is a right continuous Markov chain on finite state space $\mathbb{S}$; σ is a real number and f is a real Lipschitz continuous function on $L^2(0,1)$ satisfying for $u, v \in L^2(0,1)$ and some positive constants α, β,

$$|f(u,r)-f(v,r)|^2 \leq \alpha\|u-v\|_H^2, \ \ (5.2)$$

$$|f(u,r)|^2 \leq \beta(1+\|u\|_H^2). \quad (5.3)$$

In this example, we take $H=L^2(0,1)$ and $A=\frac{\partial^2}{\partial x^2}$ with

$$D(A)=\{y \in H, y, y'\, are\ absolutely\ continuous\ with\ y', y'' \in H. y(0)=y(1)=0\}.$$

By Govindan [16], let the function $e_n = \sqrt{2/\pi}\sin(\pi n x), n \in N, x \in [0,1]$, be orthonormal eigenvectors of A, corresponding to the eigenvalues $-\pi^2 n^2, n \in N$. For any $u \in D(A)$,

$$Au = \sum_{n=1}^{\infty} < u, e_n > Ae_n = \sum_{n=1}^{\infty} < u, e_n > (-\pi^2 n^2)\, e_n,$$

$$\begin{aligned} < u, Au > &= -\sum_{n=1}^{\infty} \pi^2 n^2 < u, e_n >< u, e_n > \\ &= -\sum_{n=1}^{\infty} \pi^2 n^2 < u, e_n >^2 \\ &\le -\pi^2 \|u\|_H^2 . \end{aligned}$$

Then, referring to (3.1), A generates a contraction semigroup $G(t)$ in $H = L^2(0,1)$, and $F = 0$, $\Sigma = \sigma f\big(y(x,t), \theta(t)\big)$. It is easy to check that conditions (5.2) and (5.3) imply that conditions (H1) and (H2) for Theorem 3.4 are satisfied. By theorem 3.4, we conclude that the problem (5.1) has a unique mild solution $y \in L^P\big(\Omega; C([0,T]; H)\big)$ for any $p > 2$.

Next, we show that the Markov process $(y(x,t), \theta(t)) \in H \times \mathbb{S}$ satisfies Feller property. By theorem 4.1, the solution of system (5.1) is

$$y_0^{y_0,i}(t) = G_t y_0 + \int_0^t G_{t-v}\sigma f\left(y_0^{y_0,i}(v), \theta_0^i(v)\right) dW(v), \qquad t \ge 0,$$

where initial values $y_0(x) \in H = L^2(0,1)$, and $\theta_0 = i \in \mathbb{S}$, both of initial data is $\mathcal{F}_0$-measurable. For any bounded continuous function $\psi: H \to H$ and for any fixed $i \in \mathbb{S}, \delta > 0$. Note that for $0 \le u \le t \le T$,

$$\begin{aligned} E\psi\left(y_0^{y_0,i}(0+\delta)\right) - E\psi\left(y_u^{y_u,i}(u+\delta)\right) = E\psi\left(y_0^{y_0,i}(0+\delta)\right) - E\psi\left(y_0^{y_0,i}(u+\delta)\right) + \\ E\psi\left(y_0^{y_0,i}(u+\delta)\right) - E\psi\left(y_u^{y_u,i}(u+\delta)\right), \qquad (5.4) \end{aligned}$$

where y_0 is $\mathcal{F}_0$-measurable, y_u is $\mathcal{F}_u$-measurable and $E\|y_0\|^2 < \infty$, $E\|y_u\|^2 < \infty$. On the one hand, by lemma 4.2 and dominated convergence theorem, we have

$$E\psi\left(y_0^{y_0,i}(u+\delta)\right) - E\psi\left(y_u^{y_u,i}(u+\delta)\right) \to 0, \quad as\ (y_u, u) \to (y_0, 0).$$

On the other hand, we obtain $y_0^{y_0,i}(0+\delta)$ convergences to $y_0^{y_0,i}(u+\delta)$ as $u \to 0$. Then using the dominated convergence theorem, we get

$$E\psi\left(y_0^{y_0,i}(0+\delta)\right) - E\psi\left(y_0^{y_0,i}(u+\delta)\right) \to 0, \quad as\ u \to 0.$$

Therefore,

$$E\psi\left(y_0^{y_0,i}(0+\delta)\right) - E\psi\left(y_u^{y_u,i}(u+\delta)\right) \to 0, \ \ as\ (y_u, u) \to (y_0, 0).$$

In other words, $E\psi\left(y_0^{y_0,i}(0+\delta)\right)$ as a function of $(y_0, 0)$ is a continuous and that is Feller property. □

6. CONCLUSIONS

This paper is devoted to study the system of SHEs-MS. This system is treated as nonlinear stochastic evolution equations, and the existence and uniqueness of mild solutions is verified through the semigroup approach. Next we investigate the Feller property of two-component Markov processes which consist of continuous dynamics and discrete events. This area is becoming increasingly useful in engineering, financial modelling, active networking, and so forth.

Some important and interesting questions can be further investigated using the results in this paper. In particular, the stability of SHEs-MS is one of the most important and interesting topics, and those investigations are in progress.

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REFERENCES

1. Anabtawi, M.J. and Sathananthan, S. (2009): Stability and Convergence results for Ito-type Parabolic Partial Differential Equations in Hilbert Spaces, Stochastic Analysis and Applications, Vol.27, No.4, pp.671-693.
2. Anabtawi, M.J. and Sathananthan, S. (2011): Ito-type Parabolic Partial Differential Equations in Hilbert Spaces via Lyapunov-like function Approach, Stochastic Analysis and Applications, Vol. 29, No.5, pp.749-768.
3. Chow, P. L. (2007): Stochastic Partial Differential Equations, Chapman&Hall/CRC Press, Taylor& Francis Group.
4. Da Prato, G., Zabczyk, j. (1992): Stochastic Equations in Infinite Dimensions, Cambridge Univ. Press, Cambridge.
5. Da Prato, G., Zabczyk, j. (1996): Ergodicity for Infinite Dimensions systems, Cambridge Univ. Press, Cambridge.
6. Hutson, V., Pym, J.S. (1980): Applications of Functional Analysis and Operator Theory, Academic Press, New York.
7. Ikeda, N., Watanabe, S. (1989): Stochastic Differential Equations and Diffusion Process, North-Holland, Amsterdam.
8. Kunita, H. (1990): Stochastic Flows and Stochastic Differential Equations, Cambridge University Press, Cambridge.
9. Luo, Q., Mao, X. (2007): Stochastic Population Dynamics under Regime Switching, J. Math. Anal. Appl.334, 69-84.
10. Luo, Q., Mao, X. (2009): Stochastic Population Dynamics under Regime Switching II, J. Math. Anal. Appl. 355, 577-593.

11. Mao, X., Yuan, C. (2006): Stochastic Differential Equations with Markovian Switching, Imperial College Press, London.
12. Piech, M.A. (1972): Diffusion semigroups on abstract Wiener space, Trans. Amer. Math. Soc., 166, 411-430.
13. Settati, A., Lahrouz, A. (2014): Stationary Distribution of Stochastic Population Systems under Regime Switching, Appl. Math. Comput. 244, 235-243.
14. Skornohod, A.V. (1989): Asymptotic Methods in the Theory of Stochastic Differential Equations, American Mathematical Society.
15. Sz.-Nagy, B., Foias, C. (1970): Harmonic Analysis of Operators on Hilbert Spaces, North-Holland, Amsterdam.
16. Govindan, T.E. (2005): Almost sure exponential stability for stochastic neutral partial functional differential equations, Stochastics, 77, 139-154. MR2151664 (2006d: 60103)
17. Zhang, Z., Tong, J., Hu. L. (2016): Long-Term Behaviour of Stochastic Interest Rate Models with Markov Switching, Insurance: Mathematics and Economics 70, 320-326.